THE SOLUTION OF THE THREE-DIMENSIONAL AXISYMMETRIC PROBLEM IN THE THEORY OF ELASTICITY BY MEANS OF LINE INTEGRALS

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In papers [1 and 2] a solution is given for the axisymmetric problem of the theory of elasticity using definite integrals of analytic functions. In this paper stresses and strains of an elastic body are presented as integrals of certain complex functions which are defined on the contour of the body and which are not boundary values of the analytic functions. With this definition the real and imaginary parts of a function are not interrelated, which considerably simplifies the solution. The expressions obtained are used to derive the integral equations of the first and second fundamental problems of the theory of elasticity. The paper considers the cases of a finite and an infinite body and also a half-space. In this case of the half-space the solution is found in closed form. A numerical solution of an example of a mixed problem is given.

1. Consider a solid elastic body formed by the rotation of a symmetrical plane figure. We assume that the figure is bounded by a simple smooth closed contour L (Fig.1). It is shown in [2] that in the case of axisymmetric deformation of such body the displacements may be expressed as follows:

$$2Gw(z,r) = -\frac{1}{\pi i}\int_{\bar t}^{t} [-\varkappa\varphi(\zeta) + (2z-\zeta)\varphi'(\zeta) + \psi(\zeta)]\frac{d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} \qquad (1.1)$$

$$2Gu(z,r) = -\frac{1}{\pi i r}\int_{\bar t}^{t} [\varkappa\varphi(\zeta) + (2z-\zeta)\varphi'(\zeta) + \psi(\zeta)]\frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} \qquad (r>0)$$

$$(t = z + ir, \qquad \varkappa = 3-4\nu)$$

Here G is the shear modulus, ν is Poisson's ratio, ζ is the affix of an arbitrary point in the region, $\varphi(\zeta)$ and $\psi(\zeta)$ are functions which are holomorphous within the region and

$$\operatorname{Re}\varphi(\zeta) = \operatorname{Re}\varphi(\bar\zeta), \quad \operatorname{Im}\varphi(\zeta) = -\operatorname{Im}\varphi(\bar\zeta),$$
$$\operatorname{Re}\psi(\zeta) = \operatorname{Re}\psi(\bar\zeta), \quad \operatorname{Im}\psi(\zeta) = -\operatorname{Im}\psi(\bar\zeta) \qquad (1.2)$$

We represent these functions by integrals of the Cauchy type

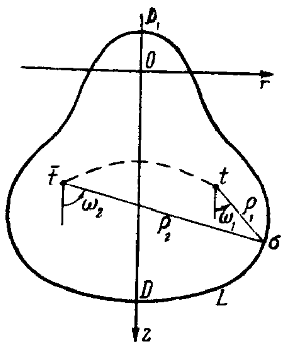

Fig. 1

$$\varphi(\zeta) = \frac{1}{2\pi i}\int_L \frac{f(\sigma)\,d\sigma}{\sigma-\zeta}, \qquad \psi(\zeta) = \frac{1}{2\pi i}\int_L \frac{F(\sigma)\,d\sigma}{\sigma-\zeta} \qquad (1.3)$$

Here σ is the affix of a point on the boundary. The direction of travel round the boundary is such that the region remains on the left. We shall take $f(\sigma)$ and $F(\sigma)$ to be functions of points on the boundary which satisfy conditions (1.2).

We substitute (1.3) into (1.1) and change the order of integration. Taking into account that

$$\frac{1}{\pi i}\int_{\bar{t}}^{t} \frac{1}{\sigma-\zeta}\frac{d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar{t})}} = \frac{1}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}$$

we obtain the following expressions for the displacements

$$2Gw(z,r) = -\frac{1}{2\pi i}\int_L [-\varkappa f(\sigma) + (2z-\sigma)f'(\sigma) + F(\sigma)]\frac{d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}} \qquad (1.4)$$

$$2Gu(z,r) = -\frac{1}{2\pi i r}\int_L [\varkappa f(\sigma) + (2z-\sigma)f'(\sigma) + F(\sigma)]\frac{(\sigma-z)\,d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}} + \frac{C}{r}$$

where

$$C = \frac{1}{2\pi i}\int_L [(\varkappa+1)f(\sigma) + F(\sigma)]\,d\sigma \qquad (1.5)$$

Here we take the branch of the root $\sqrt{(\sigma-t)(\sigma-\bar{t})}$, which becomes a real positive quantity when the point σ coincides with D. The branch line is shown by the broken line in Fig.1.

For points lying on the axis of symmetry $(r = 0)$ we must take

$$2Gw(z,0) = -\frac{1}{2\pi i}\int_L \frac{-\varkappa f(\sigma) + \sigma f'(\sigma) + F(\sigma)}{\sigma - z}\,d\sigma, \qquad 2Gu(z,0) = 0 \qquad (1.6)$$

where the right-hand side contains an integral of the Cauchy type.

Similarly, using the expressions for the stresses from [2] we obtain

$$\sigma_z(z,r) = -\frac{1}{2\pi i}\int_L [-2f'(\sigma) + (2z-\sigma)f''(\sigma) + F'(\sigma)]\frac{d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}$$

$$\sigma_\theta(z,r) = \frac{4\nu}{2\pi i}\int_L f'(\sigma)\frac{d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}} + \frac{C}{r^2} -$$

$$-\frac{1}{2\pi i r}\int_L [\varkappa f(\sigma) + (2z-\sigma)f'(\sigma) + F(\sigma)]\frac{(\sigma-z)\,d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}} \qquad (1.7)$$

$$\sigma_r(z,r) = \frac{4(1+\nu)}{2\pi i}\int_L f'(\sigma)\frac{d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}} - \sigma_z - \sigma_\theta$$

$$\tau_{rz}(z,r) = -\frac{1}{2\pi i r}\int_L [(2z-\sigma)f''(\sigma) + F'(\sigma)]\frac{(\sigma-z)\,d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}$$

$$(r > 0)$$

At $r = 0$ we have

$$\sigma_z(z,0) = -\frac{1}{2\pi i}\int_L \frac{-2f'(\sigma)+\sigma f''(\sigma)+F'(\sigma)}{\sigma-z}\,d\sigma$$

$$\sigma_r(z,0) = \sigma_\theta(z,0) = \frac{1}{4\pi i}\int_L \frac{2(1+2\nu)f'(\sigma)+\sigma f''(\sigma)+F'(\sigma)}{\sigma-z}\,d\sigma \qquad (1.8)$$

$$\tau_{rz}(z,0) = 0$$

If we draw a symmetrical, smooth arc within the region, then the intensity of forces acting on this arc from the direction of the external normal can be expressed as follows [2]:

$$p_z(z,r) = \frac{1}{2\pi ir}\frac{d}{ds}\int_L [-f(\sigma)+(2z-\sigma)f'(\sigma)+F(\sigma)]\frac{(\sigma-z)d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}$$

$$p_r(z,r) = \frac{1}{2\pi ir^2}\frac{d}{ds}\int_L [f(\sigma)+(2z-\sigma)f'(\sigma)+F(\sigma)]\Big[\sqrt{(\sigma-t)(\sigma-\bar{t})}+$$

$$+\frac{(\sigma-z)^2}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}\Big]d\sigma - \frac{\sin\alpha}{2\pi ir^2}\int_L [(3+4\nu)f(\sigma)+(2z-\sigma)f'(\sigma)+$$

$$+F(\sigma)]\frac{(\sigma-z)\,d\sigma}{\sqrt{(\sigma-t)(\sigma-\bar{t})}} - \frac{C}{r^2}\sin\alpha \qquad (1.9)$$

On the axis of symmetry

$$p_z(z,0) = \sigma_z(z,0) = -\frac{1}{2\pi i}\int_L \frac{-2f'(\sigma)+\sigma f''(\sigma)+F'(\sigma)}{\sigma-z}\,d\sigma \qquad (1.10)$$

$$p_r(z,0) = 0$$

Here ds is the differential of arc and α is the inclination of the normal to the z-axis.

We introduce the notations

$$Z = \int_0^s p_z r\,ds, \qquad R = \int_0^s p_r r^2 ds + \int_0^s Z\sin\alpha\,ds \qquad (1.11)$$

where the integration is taken along the arc from the point on the axis of symmetry.

Equalities (1.9) can be reduced to the form

$$Z = \frac{1}{2\pi i}\int_L [-f(\sigma)+(2z-\sigma)f'(\sigma)+F(\sigma)]\Big[\frac{\sigma-z}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}-1\Big]d\sigma \qquad (1.12)$$

$$R = \frac{1}{2\pi i}\int_L [f(\sigma)+(2z-\sigma)f'(\sigma)+F(\sigma)]\Big[\sqrt{(\sigma-t)(\sigma-\bar{t})}-2(\sigma-z)+$$

$$+\frac{(\sigma-z)^2}{\sqrt{(\sigma-t)(\sigma-\bar{t})}}\Big]d\sigma - \frac{4(1+\nu)}{2\pi i}\int_L f(\sigma)\left\{\int_0^s\Big[\frac{\sigma-z_1}{\sqrt{(\sigma-t_1)(\sigma-\bar{t}_1)}}-1\Big]\sin\alpha_1 ds_1\right\}d\sigma$$

The quantity $2\pi Z$ has the physical meaning of resultant of the forces acting on the surface formed by rotation of the arc about the z-axis.

2. If in equalities (1.4) and (1.12) we assume that the points t and $\bar{t}$ are points on the contour, then these equalities can be looked upon as integral equations for determining the functions $f(\sigma)$ and $F(\sigma)$. Separating the real and imaginary parts of these equalities, we obtain four independent real functions for finding two displacements or forces. Therefore to a sufficient extent two of these functions can be used quite freely.

Note that the analytic functions $\varphi(\zeta)$ and $\psi(\zeta)$ will be Kolosov-Muskhelishvili functions corresponding to the plane problem. Therefore, following the procedure of Sherman [3 and 4], we can set

$$F(\sigma) = k\overline{f(\sigma)} - \bar{\sigma} f'(\sigma) \tag{2.1}$$

where $k = 1$ in the case of the first fundamental problem and $k = -\varkappa$ in the case of the second.

Condition (1.5) gives

$$C = \frac{1}{2\pi i}\int\limits_L [(\varkappa+1) f(\sigma) + k\overline{f(\sigma)} - \bar{\sigma} f'(\sigma)]\, d\sigma$$

We make the transformation

$$\frac{1}{2\pi i}\int\limits_L \bar{\sigma} f'(\sigma)\, d\sigma = -\frac{1}{2\pi i}\int\limits_L f(\sigma)\, d\bar{\sigma} = \frac{1}{2\pi i}\int\limits_L \overline{f(\sigma)}\, d\sigma$$

As a result we obtain

$$C = \frac{1}{2\pi i}\int\limits_L [(\varkappa+1) f(\sigma) + (k-1)\overline{f(\sigma)}]\, d\sigma \tag{2.2}$$

On the basis of (1.4) we have that

$$2Gw = \frac{1}{2\pi i}\int\limits_L [\varkappa f(\sigma) - k\overline{f(\sigma)} + (\sigma + \bar{\sigma} - 2z) f'(\sigma)] \frac{d\sigma}{\sqrt{(\sigma - t)(\sigma - \bar{t})}}$$

$$2Gu = \frac{1}{2\pi i r}\int\limits_L [-\varkappa f(\sigma) - k\overline{f(\sigma)} + (\sigma + \bar{\sigma} - 2z) f'(\sigma)] \frac{(\sigma - z)\, d\sigma}{\sqrt{(\sigma - t)(\sigma - \bar{t})}} + \frac{C}{r}$$

Integrating by parts, we obtain

$$\begin{aligned}2Gw = &\frac{1}{2\pi i}\int\limits_L [\varkappa f(\sigma) - k\overline{f(\sigma)}]\, d\ln[\sqrt{(\sigma - t)(\sigma - \bar{t})} + (\sigma - z)] - \\ &-\frac{1}{2\pi i}\int\limits_L f(\sigma)\, d\left[\frac{\sigma + \bar{\sigma} - 2z}{\sqrt{(\sigma - t)(\sigma - \bar{t})}}\right]\end{aligned} \tag{2.3}$$

$$\begin{aligned}2Gu = &-\frac{1}{2\pi i r}\int\limits_L [\varkappa f(\sigma) + k\overline{f(\sigma)}]\, d\,[\sqrt{(\sigma - t)(\sigma - \bar{t})} - (\sigma - z)] - \\ &-\frac{1}{2\pi i r}\int\limits_L f(\sigma)\, d\left[\frac{(\sigma + \bar{\sigma} - 2z)(\sigma - z)}{\sqrt{(\sigma - t)(\sigma - \bar{t})}} - (\sigma + \bar{\sigma} - 2z)\right] (r > 0)\end{aligned}$$

When $r = 0$ we can apply the Sokhotskii-Plemel' formulas to (1.6) and write

$$2Gw(z_0, 0) = \frac{\varkappa - k}{2} f(z_0) + \frac{1}{2\pi i}\int\limits_L [\varkappa f(\sigma) - k\overline{f(\sigma)}]\, d\ln(\sigma - z_0) - \frac{1}{2\pi i}\int\limits_L f(\sigma)\, d\,\frac{\bar{\sigma} - z_0}{\sigma - z_0}$$

$$2Gu(z_0, 0) = 0 \tag{2.4}$$

Here z_0 is the abscissa of the point D or D_1; the integrals in the right-hand side should be taken as the principal values in the Cauchy sense. Note that when $k = -\varkappa$, equalities (2.4) can be easily transformed to the familiar Sherman-Lauricelli equation.

On the basis of Formulas (1.12) we have the following expressions for the forces

$$Z = -\frac{1}{2\pi i}\int_L [f(\sigma) - k\overline{f(\sigma)}]\, d\,[\sqrt{(\sigma - t)(\sigma - \bar{t})} - (\sigma - z)] + \qquad (2.5)$$

$$+\frac{1}{2\pi i}\int_L f(\sigma)\, d\left[\frac{(\sigma + \bar{\sigma} - 2z)(\sigma - z)}{\sqrt{(\sigma - t)(\sigma - \bar{t})}} - (\sigma + \bar{\sigma} - 2z)\right]$$

$$R = \frac{1}{2\pi i}\int_L [f(\sigma) + k\overline{f(\sigma)}]\, d\,[(\sigma - z)\sqrt{(\sigma - t)(\sigma - \bar{t})} - (\sigma - z)^2] + \frac{1}{2\pi i}\int_L f(\sigma)\, dK(\sigma, t)$$

where

$$K(\sigma, t) = (\sigma + \bar{\sigma} - 2z)\left[\sqrt{(\sigma - t)(\sigma - \bar{t})} - 2(\sigma - z) + \frac{(\sigma - z)^2}{\sqrt{(\sigma - t)(\sigma - \bar{t})}}\right] -$$

$$- 4(1 + \nu)\int_0^s [\sqrt{(\sigma - t_1)(\sigma - \bar{t}_1)} - (\sigma - z_1)] \sin\alpha_1 ds_1$$

Equalities (2.5) and (2.3) can be treated as a system of integral equations for the solution of the first and second fundamental problems of the theory of elasticity.

3. All the above formulas are valid also for an elastic space containing an axisymmetric cavity. In this case the points t and $\bar{t}$ lie outside the contour L. The branch line is drawn such that it intersects the z-axis below the cavity. For points lying on the axis of symmetry below and above the cavity we have, respectively

$$\sqrt{(\sigma - t)(\sigma - \bar{t})} = \sigma - z, \qquad \sqrt{(\sigma - t)(\sigma - \bar{t})} = -(\sigma - z)$$

Therefore Formulas (1.6) retain their form for points lying below the cavity. For points above the cavity the sign of the integral in the first equality of (1.6) must be changed. In order to satisfy the second equality we must set

$$\frac{1}{2\pi i}\int_L [(\varkappa + 1) f(\sigma) + F(\sigma)]\, d\sigma = C = 0 \qquad (3.1)$$

In evaluating the integrals we take as positive the direction of travel round the contour such that the infinite region remains on the left.

If we write the first of formulas (1.12) for an arbitrary point above the cavity, then taking into account (3.1), we obtain

$$Z_0 = -\frac{2}{2\pi i}\int_L F(\sigma)\, d\sigma = \frac{2(\varkappa + 1)}{2\pi i}\int_L f(\sigma)\, d\sigma \qquad (3.2)$$

Here $2\pi Z_0$ is the resultant, taken with opposite sign, of the forces applied to the cavity.

In tne solution of boundary-value problems we can make use of the integral equations (2.3) to (2.5). Taking into account (3.1) and (2.2) we can reduce condition (3.2) to the form

$$Z_0 = \frac{2(\varkappa + 1)}{2\pi i}\int_L f(\sigma)\, d\sigma = -\frac{2(k - 1)}{2\pi i}\int_L \overline{f(\sigma)}\, d\sigma \qquad (3.3)$$

Hence we see that the case of $k = 1$ corresponds to the action of forces in equilibrium.

4. Consider the half-space $z < z_0$. In this case in Formulas (2.3) and (2.5)

$$z = z_0, \quad t = z_0 + ir, \quad \bar{t} = z_0 - ir, \quad \sigma = z_0 + ix, \quad d\sigma = i dx, \quad \sin\alpha = 0$$

$$\sqrt{(\sigma - t)(\sigma - \bar{t})} = \begin{cases} -i\sqrt{x^2 - r^2} & \text{for} \quad -\infty < x < -r \\ \sqrt{r^2 - x^2} & \text{for} \quad -r < x < r \\ i\sqrt{x^2 - r^2} & \text{for} \quad r < x < \infty \end{cases}$$

We shall assume that as $|x| \to \infty$, $f(\sigma)$ tends to some finite value A, the same whether $x \to +\infty$ or $x \to -\infty$, and that for sufficiently large $|x|$

$$f(\sigma) = A + O(|x|^{-\mu}) \qquad (\mu > 0) \tag{4.1}$$

The integrals in (2.3) and (2.5) might be divergent, in which case they should be treated as principal values. We express the required function in the form

$$f(\sigma) = p(x) + iq(x) \tag{4.2}$$

where the right-hand side contains real functions; also, from (1.2) and (4.1)

$$p(x) = p(-x), \quad q(x) = -q(-x), \quad p(\infty) = A, \quad q(\infty) = 0 \tag{4.3}$$

In the solution of the second fundamental problem, i.e. when displacements are specified on the boundary, we set $k = -\varkappa$. Formulas (2.3) and (2.4) assume the form

$$2Gw(z_0, r) = \frac{2\varkappa}{\pi}\int_0^r p(x)\frac{dx}{\sqrt{r^2 - x^2}}, \qquad 2Gw(z_0, 0) = \varkappa p(0)$$

$$2Gu(z_0, r) = \frac{2\varkappa}{\pi r}\int_0^r q(x)\frac{x dx}{\sqrt{r^2 - x^2}}, \qquad u(z, 0) = 0 \tag{4.4}$$

These equations are easily reduced to integral equations of the Abel type, and solving these we obtain

$$p(x) = \frac{1}{\varkappa}\frac{d}{dx}\int_0^x \frac{2Gw(z_0, r)\, r dr}{\sqrt{x^2 - r^2}}, \qquad q(x) = \frac{1}{\varkappa}\frac{d}{dx}\int_0^x \frac{2Gu(z_0, r)\, x dr}{\sqrt{x^2 - r^2}} \tag{4.5}$$

$$p(0) = \frac{2G}{\varkappa} w(z_0, 0), \qquad q(0) = 0$$

We can easily confirm the validity of the latter equality by assuming $u(z_0, r)$ to satisfy the Hoelder condition in the neighborhood of the point $(z_0, 0)$.

In the solution of the first fundamental problem the stresses $\sigma_z(z_0, r)$ and $\tau_{rz}(z_0, r)$ are specified on the boundary of the half-space. Formulas (1.11) give

$$Z = \int_0^r \sigma_z(z_0, r)\, r dr, \qquad R = \int_0^r \tau_{rz}(z_0, r)\, r^2 dr \tag{4.6}$$

Setting $k = 1$ we obtain from (2.5)

$$Z = \frac{2}{\pi}\int_0^r q(x)\frac{x dx}{\sqrt{r^2 - x^2}}, \qquad R = \frac{2}{\pi}\int_0^r p(x)\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}dx \tag{4.7}$$

Solving these equations, we find that

$$q(x) = \frac{d}{dx}\int_0^x \frac{Zx\, dr}{r\sqrt{x^2 - r^2}}, \qquad p(x) = p(0) + \frac{d}{dx}\int_0^x \frac{R(r^2 - 2x^2)}{r^3\sqrt{x^2 - r^2}}dr \tag{4.8}$$

where $p(0)$ is an undetermined constant.

5. Each of the integral equations (2.3) to (2.5) can be reduced to the form

$$\int_{L'} [p(\sigma)\, dU(\sigma, t) + q(\sigma)\, dV(\sigma, t)] = W(t) \tag{5.1}$$

$$p(\sigma) = \operatorname{Re} f(\sigma), \qquad q(\sigma) = \operatorname{Im} f(\sigma)$$

Here L' is the right-hand half of the contour L and $W(t)$ is a given real function.

Altough in the derivation of (2.3) tp (2.5) it was assumed for simplicity that the finctions $f(\sigma)$ and $F(\sigma)$ were differentiable, these equalities are still valid when $f(\sigma)$ is step-wise continuous and bounded on L.

Dividing the contour L' by points $\sigma_0, \sigma_1, \ldots, \sigma_{l+1}$ into a number $(l+1)$ of sufficiently small intervals and at the ends of each interval setting $f(\sigma)$ = const, we obtain

$$\sum_{n=0}^{l} \{p_n [U(\sigma_{n+1}, t) - U(\sigma_n, t)] + q_n [V(\sigma_{n+1}, t) - V(\sigma_n, t)]\} = W(t) \qquad (5.2)$$

$$(q_0 = q_l = 0)$$

Setting $t = t_m$ successively, where $m = 1, 2, \ldots, l-1$, the point t_m being situated within the interval σ_m, σ_{m+1}, we obtain a system of linear algebraic equations in p_m and q_m.

In the solution of the second fundamental problem this system must be supplemented by two equations for points on the axis of symmetry D and D_1. These equations may be obtained from (2.4).

In the solution of the first fundamental problem one of the quantities p_m can be specified arbitrarily.

When the unknowns p_m and q_m have been found, the stresses and displacements at points inside the body can be determined from Formulas (1.7) and (1.5) in which the derivatives must be avoided by integrating by parts.

As an example the problem of the penetration of an indented conical die into an elastic half-space has been solved (Fig.2).

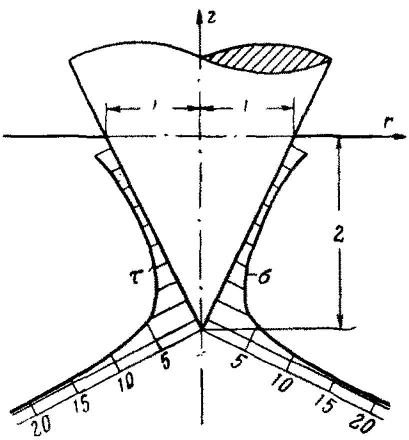

Fig. 2

The contact surface was divided into 10 equal intervals and at the mid-point of each interval the conditions $2Gw = 1$ and $u = 0$ were specified. The horizontal boundary was divided into 11 unequal intervals and at the mid-point of each the conditions Z = const and R = const were specified. The point lying on the axis of symmetry was not considered; instead p_∞ was set equal to zero. Fig.2 shows graphs of the normal tangential stresses on the contact surface, drawn from the mean values on the intervals.

The resultant of these stresses was found to be 1.295π.

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